

# Unified Approach to Universal Cloning and Phase-Covariant Cloning

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We analyze the problem of approximate quantum cloning when the quantum state is between two latitudes on the Bloch's sphere. We present an analytical formula for the optimized 1-to-2 cloning. The formula unifies the universal quantum cloning (UQCM) and the phase covariant quantum cloning.

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## I. INTRODUCTION

Recent development in quantum information have given rise to an increasing number of applications, for instance, quantum teleportation, quantum dense coding, quantum cryptography, quantum logic gates, quantum algorithms and etc [1, 2, 3, 4]. Many tasks in quantum information processing (QIP) have different properties from the classical counterpart, for example, the quantum cloning. Classically, we can duplicate (copy) any bits perfectly. In the quantum case, as shown by Wootters and Zurek [5], it is impossible to design a general machine to clone every state on the Bloch's sphere perfectly. This is called the no-cloning theorem. But such the no-cloning theorem [5] only forbids the perfect cloning. As shown by Bužek and Hillery, approximate cloning of an unknown quantum state is possible. They proposed a type of Universal Quantum Copying Machine [6] (UQCM) that clones all the state on the Bloch's sphere with the same optimal fidelity [7, 8, 9]. Subsequently, some researches have extended the UQCM to  $N$  inputs to  $M$  outputs and to  $d$ -level system [7, 10, 11]. Furthermore, studies have also been done on quantum cloning with prior information about unknown state [12, 13, 14, 15, 16, 20, 21, 23, 24], the example is the cloning of phase covariant state [12] or unknown equatorial state [21] given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle).$$

It has already been proven that the above state can be cloned with the optimal fidelity  $F = \frac{1}{2}[1 + \frac{1}{\sqrt{2}}]$  [22] and the fidelity is higher than UQCM's. This is to say, if we already have some prior information about the unknown state, we can design a better copying machine for the state. The result [12, 13, 14, 21] was subsequently extended to more general case [18, 25] and experimentally demonstrated [16, 18, 19].

The UQCM and the phase covariant cloning do not subsume each other, because one cannot be regarded as a special case of the other. In real applications of the quantum information system, we sometimes have access only to pure states distributed on a specific surface on a Bloch sphere. In this article, we study out such general situation in which the states are distributed between two latitudes on a Bloch sphere. Our result unifies the prior results pertaining to UQCM and phase covariant cloning: in particular, one could bring the two latitudes to the poles for UQCM or set the two latitudes together for phase covariant cloning.

To this end, we consider the following state:

$$|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + \sin \frac{\theta}{2}e^{i\phi}|1\rangle \quad (1)$$

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where  $\phi \in [0, 2\pi]$  and  $\theta_1 \leq \theta \leq \theta_2$ . The states we considered here are uniformly distributed between two latitudes on the Bloch sphere. When  $\theta_1 = 0$  and  $\theta_2 = \pi$ , we get the situation of the UQCM. When  $\theta_1 = \theta_2 = \frac{\pi}{2}$ , it is the phase covariant cloning. In this way, results of the UQCM and the phase covariant cloning can be unified: they are recovered as special cases of our QCM. Contrary to general perception that we can get a better QCM, we point out that this view may not always be true.

This paper is arranged as follows: After introducing some results concerning UQCM [6] and phase covariant cloning [12, 21, 22, 25], we formulate our problem in section II and present analytical results to the situation. In section III, we make detailed discussions about our  $1 \rightarrow 2$  QCM and also a qualitative discussion about the situation of  $1 \rightarrow N$  and  $M \rightarrow N$ . We end the paper with some concluding remarks.

For an arbitrary quantum state on the Bloch's sphere, we can use the following unitary transformation to get the optimal result for the cloning:

$$\begin{aligned} U : \quad & |0\rangle_a |0\rangle_b | \uparrow \rangle_x \rightarrow \sqrt{\frac{2}{3}} |0\rangle_a |0\rangle_b | \uparrow \rangle_x + \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) | \downarrow \rangle_x \\ & |1\rangle_a |0\rangle_b | \uparrow \rangle_x \rightarrow \sqrt{\frac{2}{3}} |1\rangle_a |1\rangle_b | \downarrow \rangle_x + \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) | \uparrow \rangle_x. \end{aligned} \quad (2)$$

For the state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ , after operated by the cloning operation  $U$ , we can get the density matrices  $\rho_a$  and  $\rho_b$  by taking partial trace. We then define the cloning fidelity  $F = \langle \psi | \rho_a | \psi \rangle$ . For the case of 1 to 2 UQCM, it can be proved that  $F = \frac{5}{6}$  [6].

For the phase covariant cloning there already exists a method of adjusting a parameter in the UQCM to get a better cloning fidelity [25].

## II. QUANTUM CLONING MACHINE FOR A QUBIT BETWEEN TWO LATITUDES ON THE BLOCH SPHERE

The state we wish to clone can be written as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle \quad (3)$$

where  $\phi \in [0, 2\pi]$  and

$$\theta_1 \leq \theta \leq \theta_2. \quad (4)$$

This is to say, the states we considered here are uniformly distributed in a belt between two latitudes on the Bloch sphere. We assume the following unitary transformation for our QCM:

$$\begin{aligned} U : \quad & |0\rangle_a |0\rangle_b | \uparrow \rangle_x \rightarrow \cos \alpha |0\rangle_a |0\rangle_b | \uparrow \rangle_x + \sin \alpha |\xi^+\rangle_{ab} | \downarrow \rangle_x \\ & |1\rangle_a |0\rangle_b | \uparrow \rangle_x \rightarrow \cos \beta |1\rangle_a |1\rangle_b | \downarrow \rangle_x + \sin \beta |\xi^+\rangle_{ab} | \uparrow \rangle_x \end{aligned} \quad (5)$$

where  $|\xi^+\rangle$  is defined as  $|\xi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle_{ab} + |10\rangle_{ab})$ , with  $\alpha$  and  $\beta$  being parameters that we want to determined. We restrain ourselves only to a 'symmetric' transformation and we prove its optimality below.

After transformation by the unitary operation  $U$ , we can get the following state:

$$\begin{aligned} |\psi_a\rangle |0\rangle_b | \uparrow \rangle_x \rightarrow & \cos \frac{\theta}{2} \cos \alpha |00\rangle_{ab} | \uparrow \rangle_x + \sin \alpha \cos \frac{\theta}{2} |\xi^+\rangle_{ab} | \downarrow \rangle_x \\ & + \sin \frac{\theta}{2} \cos \beta e^{i\phi} |11\rangle_{ab} | \downarrow \rangle_x + \sin \frac{\theta}{2} \sin \beta e^{i\phi} |\xi^+\rangle_{ab} | \uparrow \rangle_x. \end{aligned} \quad (6)$$

By taking partial trace, we can calculate the reduced density matrices  $\rho_a$  and  $\rho_b$  of particle a and b respectively.

$$\begin{aligned} \rho_a = \rho_b = & \left( \frac{1}{\sqrt{2}} \sin \beta \sin \frac{\theta}{2} \right)^2 |0\rangle\langle 0| + \left( \frac{1}{\sqrt{2}} \sin \alpha \cos \frac{\theta}{2} \right)^2 |1\rangle\langle 1| \\ & + \left( \cos \frac{\theta}{2} \cos \alpha \langle 0| + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} \sin \beta e^{i\phi} |1\rangle \right) \left( \cos \frac{\theta}{2} \cos \alpha \langle 0| + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} \sin \beta e^{-i\phi} \langle 1| \right) \\ & + \left( \sin \frac{\theta}{2} \cos \beta e^{i\phi} |1\rangle + \frac{1}{\sqrt{2}} \cos \frac{\theta}{2} \sin \alpha \langle 0| \right) \left( \sin \frac{\theta}{2} \cos \beta e^{-i\phi} \langle 1| + \frac{1}{\sqrt{2}} \cos \frac{\theta}{2} \sin \alpha \langle 0| \right). \end{aligned} \quad (7)$$

With the density matrices of the subsystem, we can get the fidelity

$$\begin{aligned} F &= \langle \psi | \rho_a | \psi \rangle \\ &= \cos^4 \frac{\theta}{2} \left( \frac{1}{2} + \frac{1}{2} \cos^2 \alpha \right) + \sin^4 \frac{\theta}{2} \left( \frac{1}{2} + \frac{1}{2} \cos^2 \beta \right) + \frac{1}{8} \sin^2 \theta (\sin^2 \alpha + \sin^2 \beta) + \frac{\sqrt{2}}{4} \sin^2 \theta \sin(\alpha + \beta). \end{aligned} \quad (8)$$

Averaging the fidelity over all possible angles  $\theta$ , we have [7]

$$\begin{aligned} \bar{F} &= \frac{\int_{\theta_1}^{\theta_2} F \sin \theta d\theta}{\int_{\theta_1}^{\theta_2} \sin \theta d\theta} \\ &= \frac{1}{2} + \frac{1}{6} K - P \sin(\alpha + \beta) - Q \sin^2 \alpha - R \sin^2 \beta \end{aligned} \quad (9)$$

where

$$\begin{cases} K = \cos^2 \theta_2 + \cos \theta_1 \cos \theta_2 + \cos^2 \theta_1 \\ P = \frac{\sqrt{2}}{12} K - \frac{\sqrt{2}}{4} \\ Q = \frac{1}{12} K + \frac{1}{8} (\cos \theta_1 + \cos \theta_2) \\ R = \frac{1}{12} K - \frac{1}{8} (\cos \theta_1 + \cos \theta_2) \end{cases} \quad (10)$$

and  $K, P, Q, R$  are constants with given  $\theta_1$  and  $\theta_2$ . In order to get the maximum of  $\bar{F}$ , we do a partial differentiating  $\bar{F}$  with respect to  $\alpha$  and  $\beta$ . For maximum  $\bar{F}$ , the parameters  $\alpha$  and  $\beta$  with the optimal QCM should satisfy the following equations:

$$\begin{cases} P \cos(\alpha + \beta) + Q \sin(2\alpha) = 0 \\ P \cos(\alpha + \beta) + R \sin(2\beta) = 0. \end{cases} \quad (11)$$

#### A. Solution of $\alpha$ and $\beta$ with maximum $\bar{F}$ .

With the formulation above, we can now seek the solution of  $\alpha$  and  $\beta$  with maximum  $\bar{F}$ . Consider the situation in which the state cover the whole Bloch sphere. For this situation,  $\theta_1 = 0$  and  $\theta_2 = \pi$ , and we have  $K = 1$ ,  $P = -\frac{\sqrt{2}}{6}$  and  $Q = R = \frac{1}{12}$ . We can also solve the equations (11) to get  $\cos \alpha = \cos \beta = \sqrt{\frac{2}{3}}$ . This is the well known result in UQCM.

Now the general situation, we can easily get

$$\cos(\alpha + \beta) [2QR \sin(\alpha - \beta) + P(R - Q)] = 0. \quad (12)$$

If it satisfies that

$$\left| \frac{P(Q - R)}{2QR} \right| \leq 1 \quad (13)$$

we have

$$\sin(\alpha - \beta) = \frac{P(Q - R)}{2QR}. \quad (14)$$

Then we can get the following solution

$$2\alpha = \arcsin \left[ \frac{P(Q + R)}{S} \right] + \arcsin \left[ \frac{P(Q - R)}{2QR} \right], \quad 2\beta = \arcsin \left[ \frac{P(Q + R)}{S} \right] - \arcsin \left[ \frac{P(Q - R)}{2QR} \right] \quad (15)$$

where  $S = -\sqrt{4QRP^2 + 4Q^2R^2}$ . If  $\left| \frac{P(Q - R)}{2QR} \right| > 1$ , we can get  $\cos(\alpha + \beta) = 0$  and  $\sin 2\alpha = 0, \sin 2\beta = 0$ . Since  $P \leq 0$ , there are only two cases

1. If  $|\theta_1 - \frac{\pi}{2}| \geq |\theta_2 - \frac{\pi}{2}|$ , we have  $\alpha = 0, \beta = \frac{\pi}{2}$  and  $\bar{F} = \frac{1}{2} + \frac{1}{6}K - P - R$ ;
2. if  $|\theta_1 - \frac{\pi}{2}| < |\theta_2 - \frac{\pi}{2}|$ , we have  $\alpha = \frac{\pi}{2}, \beta = 0$  and  $\bar{F} = \frac{1}{2} + \frac{1}{6}K - P - Q$ .

In summary, the mean fidelity of our QCM is:

$$\bar{F} = \begin{cases} \frac{1}{2} + \frac{1}{6}K - P \sin(\alpha + \beta) - Q \sin^2 \alpha - R \sin^2 \beta, & |T| \leq 1; \\ \frac{1}{2} + \frac{1}{6}K - P - R, & |T| > 1 \text{ and } |\theta_1 - \frac{\pi}{2}| \geq |\theta_2 - \frac{\pi}{2}|; \\ \frac{1}{2} + \frac{1}{6}K - P - Q, & |T| > 1 \text{ and } |\theta_1 - \frac{\pi}{2}| < |\theta_2 - \frac{\pi}{2}|. \end{cases} \quad (16)$$

where  $T = \frac{P(Q-R)}{2QR}$  and  $\alpha, \beta$  are given in Eq. (15).

## B. Optimization

Our QCM discussed above has a 'symmetric' form defined by Eq. (5). In this section, we will prove that the symmetric form is necessary. We consider all possible forms of quantum cloning. If the transformation U of the bases is not in such a symmetric form, we know that the reduced density matrices of the particle a and b are not equal to each other. We then get the different fidelities for particle a and b

$$F_a = \langle \psi | \rho_a | \psi \rangle, \quad F_b = \langle \psi | \rho_b | \psi \rangle.$$

For the purpose of the cloning, we can only define the fidelity as follows

$$F_U = \min(F_a, F_b) \quad (17)$$

Then we can construct another quantum cloning machine Q to satisfy:

$$\rho'_a = \rho_b, \quad \rho'_b = \rho_a \quad (18)$$

where  $\rho_a$  and  $\rho_b$  are the reduced density matrices of two particles a and b.

If we use U and Q with probability  $\frac{1}{2}$  to copy the state, the reduced density matrices of two particles will be the same  $\frac{1}{2}(\rho_a + \rho_b)$ . In this situation, the cloning fidelity  $F_Q = \langle \psi | \frac{1}{2}(\rho_a + \rho_b) | \psi \rangle$ . We have  $F_Q \geq F_U$ . Finally we get a symmetric form. It can also be said that for every possible cloning, we can always find a 'symmetric' cloning transformation that is optimal. So we need only consider the form given by Eq. (5). After getting the solution to this symmetric situation, we find the optimal QCM.

## III. SOME DISCUSSION ABOUT OUR QCM

Given  $\theta_1$  and  $\theta_2$ , we can calculate the optimal fidelity by using Eq. (16). Fig.1 presents all the situation with states uniformly distributed in any belt on the Bloch space. With observation from Fig.1 and simple derivation, we arrive at the following results:

1. If  $\theta_1 = 0, \theta_2 = \pi$ , it is the situation of the UQCM and the optimal fidelity is  $F = \frac{5}{6}$  which corresponds to points B<sub>1</sub> or B<sub>2</sub> in Fig.1;
2. If  $\theta_1 = \theta_2 = \frac{\pi}{2}$ , we encounter the situation of Phase-covariant QCM and the optimal fidelity is  $F = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$  which corresponds to the point C in Fig.1;
3. Fixed one latitude of the belt, we can set  $\theta_1$  to be constant without closing any generality, the minimum fidelity will be get at the point with  $\theta_2 = \pi - \theta_1$ . For example, fixed  $\theta_1 = \frac{\pi}{4}$ , Fig.2 draw the optimal fidelity with  $\theta_2 \in [\frac{\pi}{4}, \pi]$ . The minimum optimal fidelity obtained when  $\theta_2 = \pi - \theta_1 = \frac{3\pi}{4}$ . Contrary to one's intuition, the fidelity of cloning can rise with the area of where the unknown input state is on.

In general, we sometimes will encounter the problem of multiple cloning, i.e.  $1 \rightarrow N$  and  $M \rightarrow N$ . Here we discuss the results for  $1 \rightarrow 2$  cloning to case of  $1 \rightarrow N$  and  $M \rightarrow N$  qualitatively.

For the states  $|\psi\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} e^{i\phi} |\downarrow\rangle$ , where  $\phi \in [0, 2\pi]$  and  $\theta_1 \leq \theta \leq \theta_2$ .

We can assume [22]

$$\begin{aligned} U_{1,N} |\uparrow\rangle \otimes R &= \sum_{j=0}^{N-1} a_j |(N-j) \uparrow, j \downarrow\rangle \otimes R^j \\ U_{1,N} |\downarrow\rangle \otimes R &= \sum_{j=0}^{N-1} b_{M-1-j} |(N-1-j) \uparrow, (j+1) \downarrow\rangle \otimes R^j \end{aligned}$$

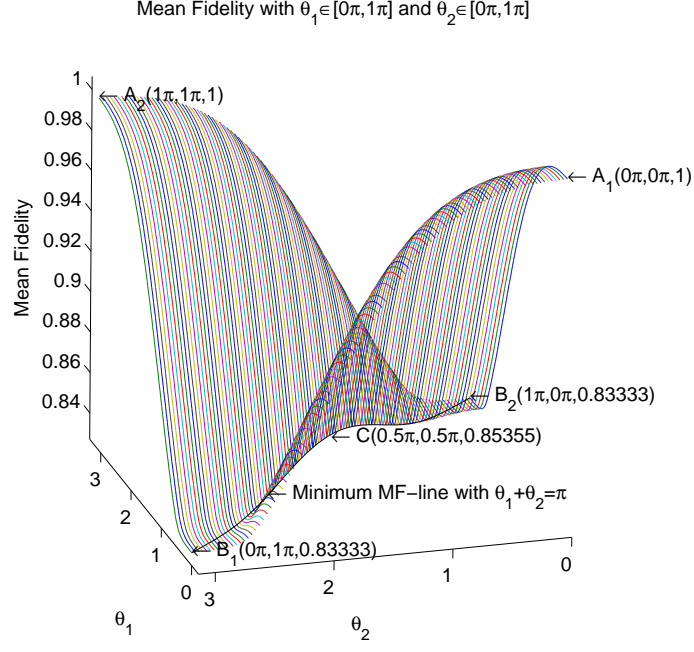


FIG. 1: The optimal fidelity of 1 to 2 cloning for states between any tow latitudes of Bloch space. Point  $B_1$  and  $B_2$  correspond to the situation of UQCM. Point  $C$  corresponds to the situation of Phase-covariant QCM. The bottom line corresponds to the situation of  $\theta_1 + \theta_2 = \pi$ .

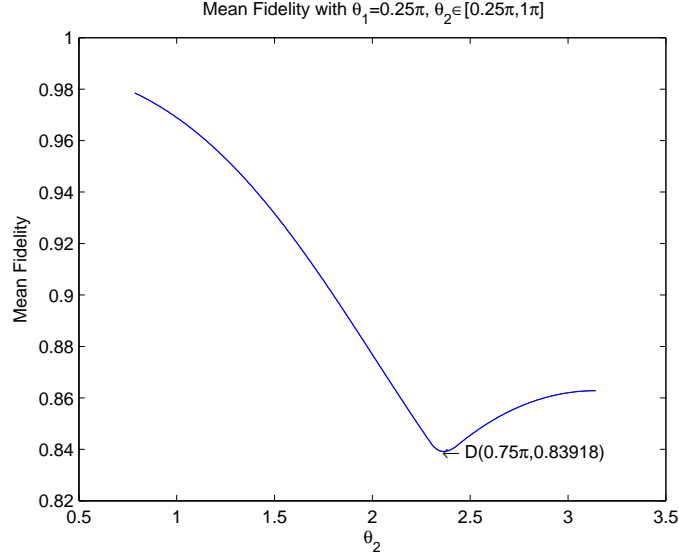


FIG. 2: The optimal fidelity with  $\theta_1 = \frac{\pi}{4}$ ,  $\theta_2 \in [\frac{\pi}{4}, \pi]$ . Point  $D$  corresponds to the minimum optimal fidelity with  $\theta_2 = \frac{3\pi}{4}$ .

where  $R$  and  $R_j$  are the auxiliary quantum system.

We know that the parameters of particle a and b are not completely independent. We must let the reduced density matrices of  $N$  particles to be the same form in order to achieve optimality, so we can assume that the cloning transformation of bases take a symmetric form.

Using the same method and defining the fidelity as  $F = \langle \psi | \rho_a | \psi \rangle$ , we get

$$\bar{F} = \frac{\int_{\theta_1}^{\theta_2} F \sin \theta d\theta}{\int_{\theta_1}^{\theta_2} \sin \theta d\theta}. \quad (19)$$

and we calculate the partial derivative es of the free parameters  $a_j$  and  $b_j$  and set them to zero, getting the following equations:

$$\frac{\partial \bar{F}}{\partial a_j} = 0, \quad \frac{\partial \bar{F}}{\partial b_j} = 0. \quad (j = 0, 1, \dots, N-1) \quad (20)$$

These equations are high-order multi-variant equations and in general the higher-order equations has no analytical solution. A similar result can be obtained for the  $M \rightarrow N$  situation.

#### IV. CONCLUDING REMARK

In summary, we have presented the quantum cloning machine for qubits uniformly distributed on a belt between two latitudes of the Bloch sphere. Previous results regards  $1 \rightarrow 2$  cloning of both the universal cloning and the phase covariant cloning can unified into a single formulism. So that the previous results are recovered as special case of the current results.

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